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Sequent calculi and decidability for intuitionistic hybrid logic

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ABSTRACT

In this paper we study the proof theory of the first constructive version of hybrid logic called Intuitionistic Hybrid Logic (IHL) in order to prove its decidability. In this perspective we propose a sequent-style natural deduction system and then the first sequent calculus for this logic. We prove its main properties like soundness, completeness and also the cut-elimination property. Finally we provide, from our calculus, the first decision procedure for IHL and then prove its decidability.

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1. Introduction

In the standard Kripke semantics for modal logics, a model is a transition system where the same formula may have different truth values at different worlds [4,9]. The hybrid logics were mainly introduced in order to express this relativity of truth [2,3] by adding to modal logics a new kind of propositional symbols called *nominals*, and also a new operator, called *satisfaction operator*, that allows one to jump to the world named by a nominal. There exist many works on hybrid logics, mainly on classical versions, about calculi, decidability and complexity [1,2,6,19].

In this work we aim at studying an intuitionistic version of hybrid logic called IHL and defined by Braüner and de Paiva [7]. It has been designed from the intuitionistic modal logic IK introduced in [17], knowing that intuitionistic modal logics have some important applications in computer science, for instance for formal verification of computer hardware [11] or definition of programming languages [10,14]. There exists a natural deduction system for IHL, extended with additional inference rules corresponding to conditions on the accessibility relation but in this logic the proof theory, through the sequent calculus formalism, and the decidability have not really been explored. There is also another constructive version of hybrid logic [13] that is based on the intuitionistic modal logic IS5 [17] and later enriched with the disjunctive connective and the constant denoting absurdity [8]. However, this logic cannot be seen as a complete hybridization of IS5 because the nominals (called *places* in the original paper) are only used with the satisfaction operator. We have recently studied proof theory for this logic by defining sequent calculi dedicated to proof and countermodel construction [12]. Thus we have given an alternative proof of decidability by proof-theoretical arguments and shown that the sequent calculus formalism is a good formalism allowing an effective management of nominals in the proof-search process. Even if IHL is also an intuitionistic hybrid logic these results cannot be directly extended for this logic.

In this paper we consider the intuitionistic hybrid logic IHL for which, as said before, there only exists a natural deduction system [7] and the decidability is still an open question. In order to solve it, we mainly propose a sequent calculus for IHL that is adapted to proof-search but also to the study of decidability. There are many works on classical versions of hybrid logics but they cannot be directly adapted in order to propose a sequent calculus allowing to show decidability in such an intuitionistic version of hybrid logic. A key point is to solve the problem of the introduction of new nominals due to some rules, that is similar to the introduction of new labels in the labelled sequent calculi of intuitionistic modal logics [17].

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Let us note that the introduction of new nominals or labels is not a problem in the case of classical modal and hybrid logics because we can define proof systems with only invertible rules that lead to terminating proof-search [5,15]. But it is a real problem for intuitionistic versions and one needs to introduce new appropriate concepts to deal with it.

In this context the main contributions of this work are: the definition of a sequent calculus for IHL, the proofs of its soundness and completeness and also of the cut-elimination property, and finally the first proof of its decidability based on this calculus. From these results we will study in next works the complexity of IHL [1] but also extensions of our sequent calculus with rules corresponding to conditions on the accessibility relations (geometric theories) like reflexivity, symmetry and transitivity, in order to obtain a modular system in which each condition on the accessibility relation has a corresponding rule and each combination of these rules is complete for the logic with the corresponding conditions.

Section 2 presents the first constructive version hybrid logic IHL [7] and its known related results. In Section 3 we give a sequent-style natural deduction system for IHL, denoted DNS_{IHL} , in order to deal with validity in IHL. It is derived from the initial natural deduction system DN_{IHL} [7]. In Section 4 we define a sequent calculus for IHL, called G_{IHL} , and then we prove its soundness and completeness. Moreover we prove that our calculus satisfies the cut-elimination property and finally we show how to derive another sequent calculus for IHL. In Section 5 we prove the decidability of IHL from our sequent calculus. The key point of the decision procedure is the use of the cut-elimination property in order to provide a suitable subformula property different from the usual one: the *quasi-subformula property*. In this context we introduce a notion of *redundancy* on cut-free derivations in our calculus such that any sequent that is valid has an irredundant proof. Then, by using the quasi-subformula property, we prove that there is no infinite proof which is not irredundant and then we provide a decision procedure for IHL and prove its decidability.

2. Intuitionistic hybrid logic

Hybrid logics are logics obtained by adding to modal logics a new kind of propositional symbols, called *nominals*, which are used to refer to specific worlds in a model and also a new operator called *satisfaction operator* that allows us to jump to the worlds named by nominals. For more details about hybrid logics see [3]. In this paper, we focus on the first constructive version of hybrid logic IHL [7].

Let Prop be a countably set of propositional symbols and Nom be a countably set of nominals that is disjoint from Prop . We use p, q, r, \dots to range over Prop and a, b, c, \dots to range over Nom . Moreover, we use $\text{Nom}(S)$ to denote the set of nominals that appear in the syntactic object S .

The formulas of IHL are given by the following grammar:

$$A ::= p \mid a \mid \perp \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Box A \mid \Diamond A \mid a : A$$

Definition 1. An IHL-Kripke model is a tuple

$$(W, \leq, \{D_w\}_{w \in W}, \{\sim_w\}_{w \in W}, \{R_w\}_{w \in W}, \{V_w\}_{w \in W})$$

with

- W is a non-empty set (of ‘worlds’) partially ordered by \leq ;
- for each $w \in W$, D_w is a non-empty set such that if $w \leq w'$ then $D_w \subseteq D_{w'}$;
- for each $w \in W$, \sim_w is an equivalence relation on D_w such that if $w \leq w'$ then $\sim_w \subseteq \sim_{w'}$;
- for each $w \in W$, R_w is a binary relation on D_w such that if $w \leq w'$ then $R_w \subseteq R_{w'}$;
- for each $w \in W$, V_w is a function that assigns to each $p \in \text{Prop}$ a subset of D_w such that if $w \leq w'$ then $V_w(p) \subseteq V_{w'}(p)$.

Moreover

- if $d \sim_w d'$, $e \sim_w e'$ and $R_w(d, e)$ then $R_w(d', e')$;
- if $d \sim_w d'$ and $d \in V_w(p)$ then $d' \in V_w(p)$.

Given an IHL-Kripke model $(W, \leq, \{D_w\}_{w \in W}, \{\sim_w\}_{w \in W}, \{R_w\}_{w \in W}, \{V_w\}_{w \in W})$ and an element $w \in W$, a w -assignment is a function which assigns to each nominal an element of D_w .

Definition 2. Let $\mathcal{M} = (W, \leq, \{D_w\}_{w \in W}, \{\sim_w\}_{w \in W}, \{R_w\}_{w \in W}, \{V_w\}_{w \in W})$ be an IHL-Kripke model, $w \in W$, g be a w -assignment, $d \in D_w$ and F be a formula, the relation $\mathcal{M}, g, w, d \models F$ is inductively defined on the structure of F as follows:

- $\mathcal{M}, g, w, d \models p$ iff $d \in V_w(p)$;
- $\mathcal{M}, g, w, d \models a$ iff $g(a) \sim_w d$;
- $\mathcal{M}, g, w, d \models \perp$ never;

- $\mathcal{M}, g, w, d \models A \wedge B$ iff $\mathcal{M}, g, w, d \models A$ and $\mathcal{M}, g, w, d \models B$;
- $\mathcal{M}, g, w, d \models A \vee B$ iff $\mathcal{M}, g, w, d \models A$ or $\mathcal{M}, g, w, d \models B$;
- $\mathcal{M}, g, w, d \models A \supset B$ iff for all $v \geq w$, if $\mathcal{M}, g, v, d \models A$ then $\mathcal{M}, g, v, d \models B$;
- $\mathcal{M}, g, w, d \models \Box A$ iff for all $v \geq w$, $d' \in D_v$, if $R_v(d, d')$ then $\mathcal{M}, g, v, d' \models A$;
- $\mathcal{M}, g, w, d \models \Diamond A$ iff there exists $d' \in D_w$ s.t. $R_w(d, d')$ and $\mathcal{M}, g, w, d' \models A$;
- $\mathcal{M}, g, w, d \models a : A$ iff $\mathcal{M}, g, w, g(a) \models A$.

Let $\mathcal{M} = (W, \leq, \{D_w\}_{w \in W}, \{\sim_w\}_{w \in W}, \{R_w\}_{w \in W}, \{V_w\}_{w \in W})$ be an IHL-Kripke model, a formula A is *valid* in \mathcal{M} , denoted $\mathcal{M} \models A$, if and only if $\mathcal{M}, g, w, d \models A$ for every $w \in W$, every w -assignment g and every $d \in D_w$. A formula is *valid* in IHL, denoted $\text{IHL} \models A$, if and only if $\mathcal{M} \models A$ for every IHL-Kripke model \mathcal{M} .

Moreover, for any formula A and any IHL-Kripke model \mathcal{M} , we have $\mathcal{M} \models A$ if and only if $\mathcal{M} \models a : A$ where $a \notin \text{Nom}(A)$. Therefore, there is no loss of generality by considering only *satisfaction statements* that are statements of the form $a : A$.

The IHL-Kripke models are different from the intuitionistic modal models by having for each world w an equivalence relation \sim_w on the set D_w of modal worlds. In order to illustrate this point let us consider the formula $a : b \vee a : \neg b$. Because of the lack of excluded middle law in an intuitionistic logic this formula cannot be valid in IHL. If we consider the satisfaction relation with nominals in the classical hybrid logic, namely $g, w, d \models a : A$ iff $g(a) = d$, then the formula is valid. But if we consider an IHL-Kripke model \mathcal{M} with two worlds w and w' and a w -assignment g such that $w < w'$, $g(a)$ and $g(b)$ are not equivalent by \sim_w and $g(a) \sim_{w'} g(b)$ then we have $\mathcal{M}, g, w, g(a) \not\models a : b \vee a : \neg b$ and thus \mathcal{M} is a countermodel of the formula. This illustrates the necessity to use an equivalence relation rather than the equality.

Proposition 1 (Monotonicity). *If $\mathcal{M}, g, w, d \models A$ and $w \leq w'$, then $\mathcal{M}, g, w', d \models A$.*

Proof. By structural induction on A . \square

Proposition 2 (Equivalence). *If $\mathcal{M}, g, w, d \models A$ and $d \sim_w d'$, then $\mathcal{M}, g, w, d' \models A$.*

Proof. By structural induction on A . \square

The first results for IHL deal with some proof-theoretical aspects based the natural deduction system DN_{IHL} [7] given in Fig. 1. No other alternative calculi like sequent calculi have been proposed and the decidability of IHL is an open question.

The main goal of this paper is to study this question and to present the first proof of decidability for IHL through a decision procedure based on a sequent calculus.

3. A sequent-style natural deduction system for IHL

In this section, we give a natural deduction system for IHL in a sequent-style that is obtained from the natural deduction system DN_{IHL} described in Fig. 1. Our main point here consists in defining a new system in order to deal with validity in IHL. It is a first step towards the sequent calculus we propose for this logic.

Definition 3 (Sequent). A sequent is a structure of the form $\Gamma \vdash C$ where Γ is a possibly empty finite multiset of satisfaction statements and C is a satisfaction statement.

In a standard way, a sequent $\Gamma \vdash C$ corresponds to the formula $(\bigwedge \Gamma) \supset C$. We use the notation $\bigwedge \Gamma$ as a shorthand for $a_1 : A_1 \wedge \dots \wedge a_k : A_k$ when $\Gamma = a_1 : A_1, \dots, a_k : A_k$. If Γ is empty, we identify $\bigwedge \Gamma$ with \top . We note $\mathcal{M}, g, w \models \Gamma$ if $\mathcal{M}, g, w, d \models \bigwedge \Gamma$ for $d \in D_w$ (the choice of d is not important because Γ contains only satisfaction statements).

Our natural deduction system DNS_{IHL} is given in Fig. 2. In fact DNS_{IHL} is nothing more than the natural deduction system DN_{IHL} with contexts. Let us note that, like in the system DN_{IHL} , we use the formulas of the form $a : \Diamond c$ to represent the accessibility relation. We can easily see that $\mathcal{M}, g, w, d \models a : \Diamond c$ if and only if $R(g(a), g(c))$.

Let us consider the formula $F = a : ((A \vee B \supset c : C) \supset (A \supset c : C))$. A proof of this formula in DNS_{IHL} is

$$\begin{array}{c}
 \frac{\frac{\frac{}{a : (A \vee B \supset c : C), a : A \vdash a : (A \vee B \supset c : C)}{a : (A \vee B \supset c : C), a : A \vdash a : (A \vee B \supset c : C)} [id] \quad \frac{\frac{}{a : (A \vee B \supset c : C), a : A \vdash a : A} [id] \quad \frac{}{a : (A \vee B \supset c : C), a : A \vdash a : (A \vee B)} [\vee_1]}{a : (A \vee B \supset c : C), a : A \vdash a : (A \vee B \supset c : C)} [\supset_E]}{a : (A \vee B \supset c : C), a : A \vdash a : c : C} [\supset_I] \\
 \frac{}{a : (A \vee B \supset c : C) \vdash a : (A \supset c : C)} [\supset_I] \\
 \frac{}{\vdash a : ((A \vee B \supset c : C) \supset (A \supset c : C))} [\supset_I]
 \end{array}$$

$\frac{a:A \quad a:B}{a:A \wedge B} [\wedge_I]$		$\frac{a:A \wedge B}{a:A} [\wedge_E^1]$	$\frac{a:A \wedge B}{a:B} [\wedge_E^2]$
$\frac{a:A}{a:A \vee B} [\vee_I^1] \quad \frac{a:B}{a:A \vee B} [\vee_I^2]$		$\frac{[a:A] \quad \vdots \quad C \quad [a:B] \quad \vdots \quad C}{C} [\vee_E]$	
$\frac{[a:A] \quad \vdots \quad a:B}{a:A \supset B} [\supset_I]$		$\frac{a:A \supset B \quad a:A}{a:B} [\supset_E]$	
$\frac{a:A}{c:a:A} [:I]$		$\frac{c:a:A}{a:A} [:E]$	
$\frac{e:A \quad a:\diamond e}{a:\diamond A} [\diamond_I]$		$\frac{a:\diamond A \quad [c:A][a:\diamond c] \quad \vdots \quad C}{C} [\diamond_E^*]$	
$\frac{[a:\diamond c] \quad \vdots \quad c:A}{a:\Box A} [\Box_I^{**}]$		$\frac{a:\Box A \quad a:\diamond e}{e:A} [\Box_E]$	
$\frac{a:\perp}{C} [\perp_E] \quad \frac{}{a:a} [ref]$		$\frac{a:c \quad a:A}{c:A} [nom]$	

* c does not occur in $a:\diamond A$, in C , or in any undischarged assumption other than the specified occurrences of $c:A$ and $a:\diamond c$.
 ** c does not occur in $a:\Box A$ or in any undischarged assumption other than the specified occurrences of $a:\diamond c$.

Fig. 1. The natural deduction system DN_{IHL} .

In order to illustrate the differences with the initial system DN_{IHL} we also give the proof of F in DN_{IHL} :

$$\frac{\frac{\frac{[a:A]}{a:A \vee B} [\vee_I^1]}{[a:A \vee B \supset c:C]} [\supset_E]}{c:C} [\supset_I] \quad \frac{}{a:A \supset c:C} [\supset_I] \quad \frac{}{a:((A \vee B \supset c:C) \supset (A \supset c:C))} [\supset_I]$$

Let us show now that our sequent-style natural deduction system DNS_{IHL} is sound and complete.

Theorem 1 (Soundness). *If a sequent \mathcal{S} has a proof in DNS_{IHL} then it is valid in IHL.*

Proof. For every rule we suppose that its conclusion is not valid and prove that one of its premises is also not valid.

Here, we only show the case $[\Box_I]$. Let $\mathcal{S} = \Gamma \vdash a : \Box A$ be a sequent that is not valid and let $\mathcal{M} = (W, \leq, \{D_w\}_{w \in W}, \{\sim_w\}_{w \in W}, \{R_w\}_{w \in W}, \{V_w\}_{w \in W})$ be a countermodel of \mathcal{S} . Then, there exist $w_0 \in W$ and a w_0 -assignment g such that $\mathcal{M}, g, w_0 \models \Gamma$ and $\mathcal{M}, g, w_0 \not\models a : \Box A$. Since $\mathcal{M}, g, w_0 \not\models a : \Box A$, we know that there exist $w_1 \geq w_0$ and $d \in D_{w_1}$ such that $R_{w_1}(g(a), d)$ and $\mathcal{M}, g, w_1, d \not\models A$. Let c be a nominal not in $Nom(\mathcal{S})$, we define the w_1 -assignment g' by $g'(c) = d$ and for any nominal b , different from c , $g'(b) = g(b)$. By Proposition 1, $\mathcal{M}, g', w_1 \models \Gamma$ holds and as $R_{w_1}(g(a), d)$ holds, we have $\mathcal{M}, g', w_1 \models a : \diamond c$. As we have $\mathcal{M}, g, w_1, d \not\models A$ and $\mathcal{M}, g, w_1 \not\models c : A$ we deduce that \mathcal{M} is a countermodel of $\Gamma, a : \diamond c \vdash c : A$. \square

Theorem 2 (Completeness). *If a sequent \mathcal{S} is valid in IHL then it has a proof in DNS_{IHL} .*

$\frac{}{\Gamma \vdash a : a} [\text{ref}]$	$\frac{}{\Gamma, a : A \vdash a : A} [\text{id}]$
$\frac{\Gamma \vdash a : \perp}{\Gamma \vdash C} [\perp_E]$	$\frac{\Gamma \vdash a : c \quad \Gamma \vdash a : A}{\Gamma \vdash c : A} [\text{nom}]$
$\frac{\Gamma \vdash a : A \quad \Gamma \vdash a : B}{\Gamma \vdash a : A \wedge B} [\wedge_I]$	$\frac{\Gamma \vdash a : A \wedge B}{\Gamma \vdash a : A} [\wedge_E^1]$
	$\frac{\Gamma \vdash a : A \wedge B}{\Gamma \vdash a : B} [\wedge_E^2]$
$\frac{\Gamma \vdash a : A}{\Gamma \vdash a : A \vee B} [\vee_I^1]$	$\frac{\Gamma \vdash a : B}{\Gamma \vdash a : A \vee B} [\vee_I^2]$
$\frac{\Gamma \vdash a : A \vee B \quad \Gamma, a : A \vdash C \quad \Gamma, a : B \vdash C}{\Gamma \vdash C} [\vee_E]$	
$\frac{\Gamma, a : A \vdash a : B}{\Gamma \vdash a : A \supset B} [\supset_I]$	$\frac{\Gamma \vdash a : A \supset B \quad \Gamma \vdash a : A}{\Gamma \vdash a : B} [\supset_E]$
$\frac{\Gamma \vdash a : A}{\Gamma \vdash c : a : A} [:_I]$	$\frac{\Gamma \vdash c : a : A}{\Gamma \vdash a : A} [:_E]$
$\frac{\Gamma \vdash c : A \quad \Gamma \vdash a : \diamond c}{\Gamma \vdash a : \diamond A} [\diamond_I]$	$\frac{\Gamma \vdash a : \diamond A \quad \Gamma, a : \diamond c, c : A \vdash C}{\Gamma \vdash C} [\diamond_E^*]$
$\frac{\Gamma, a : \diamond c \vdash c : A}{\Gamma \vdash a : \Box A} [\Box_I^{**}]$	$\frac{\Gamma \vdash a : \Box A \quad \Gamma \vdash a : \diamond c}{\Gamma \vdash c : A} [\Box_E]$

* c does not occur in $a : \diamond A$, in C or in any assumption in Γ .
 ** c does not occur in $a : \Box A$ or in any assumption in Γ .

Fig. 2. The natural deduction system DNS_{IHL} .

Proof. Completeness is obtained from the system DN_{IHL} by using the approach of [18]. Intuitively, the open assumptions in a derivation tree in DN_{IHL} are represented in the left-hand side of the corresponding sequent. We see that if we define a natural deduction system similar to DNS_{IHL} where we only replace the *discharge* of only one assumption with the discharge of all the assumptions of the same form (*complete discharge convention*), then we obtain a system equivalent to DN_{IHL} . For example the rule $[\supset_I]$ becomes

$$\frac{\Gamma, a : A \vdash a : B}{\Gamma' \vdash a : A \supset B} [\supset_I]$$

where $\Gamma' = \Gamma \setminus \{a : A\}$ (there is no occurrence of $a : A$ in Γ'). Then in order to prove the completeness of DNS_{IHL} , we only have to show that if a sequent has a derivation in the previous system then it has a derivation in DNS_{IHL} . \square

4. Sequent calculi for IHL

In this section, we propose two sequent calculi for IHL. The main one is called G_{IHL} and we prove its soundness and completeness by showing that a sequent is derivable in G_{IHL} if and only if it is derivable in DNS_{IHL} . Moreover we show that this calculus has the cut-elimination property and that we can derive another sequent calculus, called G_{IHL}^2 , without equivalence conditions like in the first one.

4.1. The sequent calculus G_{IHL}

We observe that even if there exist works on the design of sequent calculi in some classical hybrid logics [16] we cannot follow a similar approach in the case of the intuitionistic IHL logic. In our work we consider a sequent structure that contains only satisfaction statements because it easily allows us to *absorb* the structural rules in the axioms, logical and modal rules. Moreover, as in [7] the premises and the conclusion of each rule are satisfaction statements, we can relate our calculi construction with the initial systems provided for IHL. We observe that it facilitates the study of relationships between the cut-elimination and the normalization like in the case of intuitionistic logic [18]. Let us recall that a proof of normalization for DN_{IHL} is given in [7].

The *principal formula* of a rule application is defined to be any formula which is introduced by that rule except the cases of $[\supset_L]$ and $[\Box_L]$ where the principal formulas are respectively $a : A \supset B$ and $a : \Box A$.

$\frac{}{\Gamma, a : p \vdash a' : p} [id^*](a \sim a')$	$\frac{}{\Gamma, a : \perp \vdash C} [\perp]$	$\frac{}{\Gamma \vdash a' : a} [ref](a \sim a')$
$\frac{\Gamma, a : A, a : B \vdash C}{\Gamma, a : A \wedge B \vdash C} [\wedge_L]$	$\frac{\Gamma \vdash a : A \quad \Gamma \vdash a : B}{\Gamma \vdash a : A \wedge B} [\wedge_R]$	
$\frac{\Gamma \vdash a : A}{\Gamma \vdash a : A \vee B} [\vee_{R1}]$	$\frac{\Gamma \vdash a : B}{\Gamma \vdash a : A \vee B} [\vee_{R2}]$	$\frac{\Gamma, a : A \vdash C \quad \Gamma, a : B \vdash C}{\Gamma, a : A \vee B \vdash C} [\vee_L]$
$\frac{\Gamma, a : A \supset B \vdash a : A \quad \Gamma, a : B \vdash C}{\Gamma, a : A \supset B \vdash C} [\supset_L]$	$\frac{\Gamma, a : A \vdash a : B}{\Gamma \vdash a : A \supset B} [\supset_R]$	
$\frac{\Gamma, a : A \vdash C}{\Gamma, c : a : A \vdash C} [:_L]$	$\frac{\Gamma \vdash a : A}{\Gamma \vdash c : a : A} [:_R]$	
$\frac{\Gamma, a' : \diamond c, a : \Box A, c : A \vdash C}{\Gamma, a' : \diamond c, a : \Box A \vdash C} [\Box_L](a \sim a')$	$\frac{\Gamma, a : \diamond c \vdash c : A}{\Gamma \vdash a : \Box A} [\Box_R^{***}]$	
$\frac{\Gamma, a : \diamond c, c : A \vdash C}{\Gamma, a : \diamond A \vdash C} [\diamond_L^{**}]$	$\frac{\Gamma, a' : \diamond c \vdash c : A}{\Gamma, a' : \diamond c \vdash a : \diamond A} [\diamond_R](a \sim a')$	
$\frac{\Gamma \vdash a : A \quad \Gamma, a : A \vdash C}{\Gamma \vdash C} [cut]$		
<p>* p is atomic. ** c does not occur in $\Gamma, a : \diamond A \vdash C$ and $A \notin \text{Nom}$. *** c does not occur in $\Gamma \vdash a : \Box A$.</p>		

Fig. 3. The sequent calculus G_{IHL} .

We call *derivation* of a sequent S in G_{IHL} any tree labelled with sequents such that the root node is labelled with S and the labels at the immediate successors of a node n are the premises of a rule of G_{IHL} having the label at n as conclusion.

A sequent S has a *proof* in G_{IHL} , denoted $\vdash_{G_{IHL}} S$, if and only if S has a finite derivation in G_{IHL} where any leaf node is labelled with an axiom. Moreover we write $\vdash_{G_{IHL}}^n S$ if S has a proof in G_{IHL} of depth smaller or equal to n .

Let $S = \Gamma \vdash C$ be a sequent and R be the relation on $\text{Nom}(S)$ defined by: aRb if and only if $a : b$ is an element of Γ . We note \sim the reflexive, transitive, symmetric closure of R . When we associate a condition of the form $a \sim b$ to a rule, this means that \sim is the reflexive, transitive, symmetric closure of the relation obtained from the conclusion of this rule. It is easy to see that the problem of checking conditions of this form is decidable.

The rules and axioms of G_{IHL} are given in Fig. 3. Our approach is similar to the one used in the context of intuitionistic logic that leads to the calculus $G3i$ from the calculus IJ by absorbing weakening and contraction into the axioms and the logical rules (see [18]). There are conditions of the form $a \sim b$ associated to some axioms and rules of G_{IHL} that are due to the absorption of the rules $[nom]$ of DNS_{IHL} .

We illustrate the use of G_{IHL} by giving a proof of $a : \Box(b \supset c) \vdash a : \diamond b \supset \diamond c$

$$\begin{array}{c}
\frac{}{a : \Box(b \supset c), b : (b \supset c), a : \diamond b \vdash b : b} [ref] \quad \frac{}{a : \Box(b \supset c), a : \diamond b, d : b, b : c \vdash b : c} [ref] \\
\hline
\frac{}{a : \Box(b \supset c), b : (b \supset c), a : \diamond b \vdash b : c} [\supset_L] \\
\frac{}{a : \Box(b \supset c), b : (b \supset c), a : \diamond b \vdash a : \diamond c} [\diamond_R] \\
\frac{}{a : \Box(b \supset c), a : \diamond b \vdash a : \diamond c} [\Box_L] \\
\hline
\frac{}{a : \Box(b \supset c) \vdash a : \diamond b \supset \diamond c} [\supset_R]
\end{array}$$

Let us note that G_{IHL} is sound and complete without the restriction on $[id]$ that the principal formula must be atomic. However, without this restriction, G_{IHL} fails some properties necessary in our approach to prove the cut-elimination property.

Theorem 3 (Soundness). *If a sequent S has a proof in G_{IHL} then it has a proof in DNS_{IHL} .*

Proof. By induction on the structure of the proof of S in G_{IHL} . \square

Theorem 4 (Completeness). *If a sequent \mathcal{S} has a proof in DNS_{IHL} then it has a proof in G_{IHL} .*

Proof. By induction on the structure of the proof of \mathcal{S} in DNS_{IHL} . \square

We show now that G_{IHL} has the cut-elimination, namely if a sequent \mathcal{S} is provable in G_{IHL} then there exists a proof of \mathcal{S} in G_{IHL} without [cut].

4.2. Cut-elimination in G_{IHL}

The cut-elimination is one of the most important property of a sequent calculus and it generally results in the (quasi-)subformula property: in any proof of a sequent \mathcal{S} , only the (quasi-)subformulas of the formulas of \mathcal{S} appear in this proof.

Let us recall the notion of *depth-preserving admissibility*.

A rule $[R]$ is said to be *admissible* for a calculus \mathcal{C} , iff for all instances $\frac{H_1 \dots H_k}{C} [R]$ of $[R]$, if for all $i \in [1, k] \vdash_{\mathcal{C}} H_i$, then $\vdash_{\mathcal{C}} C$.

A rule $[R]$ is said to be *depth-preserving admissible* for \mathcal{C} , iff for all n , if for all $i \in [1, k] \vdash_{\mathcal{C}}^n H_i$, then $\vdash_{\mathcal{C}}^n C$.

Before to prove the cut-elimination property we show the depth-preserving admissibility properties of weakening and contraction in G_{IHL}^- that is the sequent calculus G_{IHL} without [cut]. We can also show the corresponding size-preserving admissibility properties.

Proposition 3 (Weakening). *If $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma \vdash C$ then $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : A \vdash C$.*

Proof. Let \mathcal{D} be a proof of $\Gamma \vdash C$ in G_{IHL}^- . By adding the formula $a : A$ to each sequent context in \mathcal{D} , we obtain a proof of $\Gamma, a : A \vdash C$ that has the same depth. \square

The following proposition is used to prove the depth-preserving admissibility of contraction. It is similar to the inversion lemma given in [18], knowing that for some rules of G_{IHL}^- , if the conclusion has a proof of a depth n then some of its premises has a proof of a depth smaller or equal to n .

Proposition 4 (Inversion lemma).

1. If $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : A \wedge B \vdash C$ then $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : A, a : B \vdash C$.
2. If $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : A_1 \vee A_2 \vdash C$ then $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : A_i \vdash C$, for $i = 1, 2$.
3. If $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma \vdash a : A_1 \wedge A_2$ then $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma \vdash a : A_i$, for $i = 1, 2$.
4. If $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma \vdash a : A \supset B$ then $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : A \vdash a : B$.
5. If $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : A \supset B \vdash C$ then $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : B \vdash C$.
6. If $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, c : a : A \vdash C$ then $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : A \vdash C$.
7. If $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma \vdash c : a : A$ then $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma \vdash a : A$.
8. If $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma \vdash a : \Box A$ then $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : \Diamond c \vdash c : A$ ($c \notin \text{Nom}(\Gamma \vdash a : \Box A)$).
9. If $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : \Diamond A \vdash C$ then $\vdash_{\text{G}_{\text{IHL}}^-}^n \Gamma, a : \Diamond c, c : A \vdash C$ ($c \notin \text{Nom}(\Gamma, a : \Diamond A \vdash C)$).

Proof. By induction on n . Here we only develop the case of 8.

- If $n = 0$ then $\Gamma \vdash a : \Box A$ is an axiom instance. Thus there is a formula of the form $d : \perp$ in Γ and $\vdash_{\text{G}_{\text{IHL}}^-}^0 \Gamma, a : \Diamond c \vdash c : A$ holds.
- Let us assume that $\vdash_{\text{G}_{\text{IHL}}^-}^{n+1} \Gamma \vdash a : \Box A$ by a derivation \mathcal{D} . If $a : \Box A$ is not principal in the last rule applied in \mathcal{D} , then by applying induction hypothesis to the premise(s) and using the same rule we have $\vdash_{\text{G}_{\text{IHL}}^-}^{n+1} \Gamma, a : \Diamond c \vdash c : A$.

Otherwise, $a : \Box A$ is principal and \mathcal{D} ends with

$$\frac{\Gamma, a : \Diamond c \vdash c : A}{\Gamma \vdash a : \Box A} [\Box_R^*]$$

By taking the immediate subdeduction of the premise we have $\vdash_{\text{G}_{\text{IHL}}^-}^{n+1} \Gamma, a : \Diamond c \vdash c : A$. \square

In order to prove the depth-preserving admissibility of contraction, we need to prove that if a sequent S has a proof in G_{IHL}^- of a depth n then the sequent obtained from S by renaming some nominals has a proof in G_{IHL}^- of a depth smaller or equal to n .

Definition 4. A renaming function f is a function from N to M where $N, M \subset \text{Nom}$. It is inductively extended to the formulas having nominals in N as follows:

- $f(p) = p$ where $p \in \text{Prop} \cup \{\perp\}$;
- $f(A \otimes B) = f(A) \otimes f(B)$ where $\otimes \in \{\wedge, \vee, \supset\}$;
- $f(\boxtimes A) = \boxtimes f(A)$ where $\boxtimes \in \{\square, \diamond\}$;
- $f(a : A) = f(a) : f(A)$.

We use the notation $f(\Gamma)$ for $f(a_1 : A_1), \dots, f(a_k : A_k)$ when $\Gamma = a_1 : A_1, \dots, a_k : A_k$. Moreover, the notation $f(\Gamma \vdash C)$ corresponds to $f(\Gamma) \vdash f(C)$.

Proposition 5 (Renaming). Let S be a sequent and $f : \text{Nom}(S) \rightarrow M$ be a renaming function. If $\vdash_{G_{\text{IHL}}^-}^n S$ then $\vdash_{G_{\text{IHL}}^-}^n f(S)$.

Proof. It is sufficient to prove by induction on n that if $\vdash_{G_{\text{IHL}}^-}^n S$ then $\vdash_{G_{\text{IHL}}^-}^n S[c/a]$ for any c and a of Nom , where $S[c/a]$ denotes the renaming of a by c in S . \square

The next propositions correspond to the depth-preserving admissibility of contraction.

Proposition 6 (Contraction). If $\vdash_{G_{\text{IHL}}^-}^n \Gamma, a : A, a : A \vdash C$ then $\vdash_{G_{\text{IHL}}^-}^n \Gamma, a : A \vdash C$.

Proof. By induction on n .

- If $n = 0$ then $\Gamma, a : A, a : A \vdash C$ is an instance of an axiom. Thus, it is easy to see that $\Gamma, a : A \vdash C$ is also an instance of the same axiom.
- Let us assume that $\vdash_{G_{\text{IHL}}^-}^{n+1} \Gamma, a : A, a : A \vdash C$ by a derivation \mathcal{D} . If the last rule applied in \mathcal{D} does not modify the two occurrences of $a : A$ then by induction hypothesis on the premise(s) and using the same rule, we have $\vdash_{G_{\text{IHL}}^-}^{n+1} \Gamma, a : A \vdash C$.

Otherwise we have to distinguish the cases of this last rule. Here, we only develop the case of $[\diamond_L]$ ($A \equiv \diamond B$):

$$\frac{\Gamma, a : \diamond B, a : \diamond c, c : B \vdash C}{\Gamma, a : \diamond B, a : \diamond B \vdash C} [\diamond_L]$$

By Proposition 4 we have $\vdash_{G_{\text{IHL}}^-}^n \Gamma, a : \diamond c, a : \diamond c', c : B, c' : B \vdash C$. Then by Proposition 5, $\vdash_{G_{\text{IHL}}^-}^n \Gamma, a : \diamond c, a : \diamond c, c : B, c : B \vdash C$ holds (c and c' are new nominals).

Finally by induction hypothesis applied twice we have $\vdash_{G_{\text{IHL}}^-}^n \Gamma, a : \diamond c, c : B \vdash C$ and then $\vdash_{G_{\text{IHL}}^-}^{n+1} \Gamma, a : \diamond B \vdash C$ is obtained using $[\diamond_L]$. \square

Now, we give a proposition stronger than the depth-preserving admissibility of contraction. However, it does not cover all satisfaction statements. This proposition is useful for the proof of cut-elimination.

Proposition 7. If $\vdash_{G_{\text{IHL}}^-}^n \Gamma, a : A, a' : A \vdash C$ and $a \sim a'$ in $\Gamma, a : A \vdash C$, then $\vdash_{G_{\text{IHL}}^-}^n \Gamma, a : A \vdash C$.

Proof. By induction on n , similarly to the proof of Proposition 6. \square

The two following propositions are used in the proof of cut-elimination.

Proposition 8. Let $\Gamma \vdash C$ be a sequent and $a, a' \in \text{Nom}(\Gamma \vdash C)$ such that $a \sim a'$. If $\vdash_{G_{\text{IHL}}^-}^n \Gamma, a' : a \vdash C$ then $\vdash_{G_{\text{IHL}}^-}^n \Gamma \vdash C$.

Proof. By induction on n . \square

Proposition 9. *If $\vdash_{\mathbf{G}_{\text{IHL}}}^n \Gamma \vdash a : A$ and $a \sim a'$, then $\vdash_{\mathbf{G}_{\text{IHL}}}^n \Gamma \vdash a' : A$.*

Proof. By induction on n . \square

Let \mathcal{D} be a proof of $\Gamma \vdash C$, we denote $\mathcal{D}[\Gamma']$ the proof of $\Gamma, \Gamma' \vdash C$ obtained from \mathcal{D} by applying depth-preserving admissibility of weakening.

Theorem 5 (Cut-elimination). *If a sequent S has a proof in \mathbf{G}_{IHL} then it has a proof without $[\text{cut}]$.*

Proof. To prove the cut-elimination property, we use a variant of Gentzen's original proof of this property for classical and intuitionistic logic [18]. This proof consists in transforming the applications of cut rules to applications of cut rules on smaller formulae or applications of less heights. It suffices to assume that $\vdash_{\mathbf{G}_{\text{IHL}}}^- \Gamma \vdash a : A$ and $\vdash_{\mathbf{G}_{\text{IHL}}}^- \Gamma, a : A \vdash C$ and to prove that $\vdash_{\mathbf{G}_{\text{IHL}}}^- \Gamma \vdash C$ by induction on the structure of A and on the cut depth. Let \mathcal{D}_1 be a proof of $\Gamma \vdash a : A$ in $\mathbf{G}_{\text{IHL}}^-$ of depth n_1 and \mathcal{D}_2 be a proof of $\Gamma, a : A \vdash C$ in $\mathbf{G}_{\text{IHL}}^-$ of depth n_2 , there are four main cases to consider:

1. $\Gamma \vdash a : A$ is an instance of an axiom;
2. $\Gamma, a : A \vdash C$ is an instance of an axiom;
3. $\Gamma \vdash a : A$ and $\Gamma, a : A \vdash C$ are not axiom instances and $a : A$ is not principal in the last rule of \mathcal{D}_1 or \mathcal{D}_2 ;
4. $a : A$ is principal for both last rules of \mathcal{D}_1 and \mathcal{D}_2 .

Case 1. $\Gamma \vdash a : A$ is an axiom instance.

1(a). If $\Gamma \vdash a : A$ is an instance of $[\text{id}]$, then it is of the form $\Gamma', a' : p \vdash a : p$ and $\Gamma, a : A \vdash C$ is of the form $\Gamma', a' : p, a : p \vdash C$ where $a \sim a'$ in all these sequents. By Proposition 7, we have if $\vdash_{\mathbf{G}_{\text{IHL}}}^n \Gamma, a' : p, a : p \vdash C$ with $a \sim a'$, then $\vdash_{\mathbf{G}_{\text{IHL}}}^n \Gamma, a' : p \vdash C$. Thus we obtain $\vdash_{\mathbf{G}_{\text{IHL}}}^{n_2} \Gamma, a' : p \vdash C$.

1(b). If $\Gamma \vdash a : A$ is an instance of $[\perp]$, then it is of the form $\Gamma', a' : \perp \vdash a : A$. Thus, $\Gamma \vdash C$ is of the form $\Gamma', a' : \perp \vdash C$ and it is an instance of $[\perp]$.

1(c). If $\Gamma \vdash a : A$ is an instance of $[\text{ref}]$, then it is of the form $\Gamma \vdash a : a'$ and $\Gamma, a : A \vdash C$ is of the form $\Gamma, a : a' \vdash C$ where $a \sim a'$ in these sequents. Using Proposition 8, we obtain $\vdash_{\mathbf{G}_{\text{IHL}}}^- \Gamma \vdash C$.

Case 2. $\Gamma, a : A \vdash C$ is an axiom instance.

2(a). If $\Gamma, a : A \vdash C$ is an instance of $[\text{id}]$. If it is of the form $\Gamma, a : p \vdash a' : p$, then $\Gamma \vdash a : A$ is of the form $\Gamma \vdash a : p$ with $a \sim a'$. Using Proposition 9, we deduce that $\vdash_{\mathbf{G}_{\text{IHL}}}^- \Gamma \vdash a' : p$. Now, we study the case when $\Gamma, a : A \vdash C$ is of the form $\Gamma, a : A, a' : p \vdash a'' : p$ with $a' \sim a''$. If $a' \sim a''$ in $\Gamma \vdash C$, then $\vdash_{\mathbf{G}_{\text{IHL}}}^- \Gamma \vdash C$ holds. Otherwise, $a : A$ is of the form $a : c$. If $\Gamma \vdash a : c$ is an axiom instance, then it is of the form $\Gamma', d : \perp \vdash a : c$ and we deduce that $\Gamma \vdash C$ is an instance of $[\perp]$. Now we distinguish the cases where the last rule applied in \mathcal{D}_1 is $[\supset_L]$ or not. If the last rule applied in \mathcal{D}_1 is $[\supset_L]$ then it is of the form:

$$\frac{\begin{array}{c} \mathcal{D}'_1 \\ \Gamma \vdash e : B \end{array} \quad \begin{array}{c} \mathcal{D}''_1 \\ \Gamma', e : D \vdash a : c \end{array}}{\Gamma', e : B \supset D \vdash a : c} [\supset_L]$$

Since $\Gamma, a : c \vdash C$ is an instance of $[\text{id}]$, we can easily see that $\Gamma', e : D, a : c \vdash C$ is also an instance of $[\text{id}]$. Using the induction hypothesis, we obtain a proof of $\Gamma \vdash C$ as follows:

$$\frac{\begin{array}{c} \mathcal{D}'_1 \\ \Gamma \vdash e : B \end{array} \quad \frac{\begin{array}{c} \mathcal{D}''_1 \\ \Gamma', e : D \vdash a : c \end{array} \quad \frac{}{\Gamma', e : D, a : c \vdash C} [\text{id}]}{\Gamma', e : D \vdash C} [\text{cut}]}{\Gamma \vdash C} [\supset_L]$$

with $[\text{cut}]$ that corresponds to the application of the induction hypothesis.

Otherwise, the last rule applied in \mathcal{D}_1 is not $[\supset_L]$. We consider only the case where the last rule of \mathcal{D}_1 is a two-premises rule (the case of one-premise rule being simpler):

$$\frac{\mathcal{D}'_1 \quad \mathcal{D}''_1}{\Gamma' \vdash a : c \quad \Gamma'' \vdash a : c} [R]$$

It is easy to see that $\Gamma', a : c \vdash C$ and $\Gamma'', a : c \vdash C$ are instances of $[id]$. By applying the induction hypothesis, we obtain a proof of $\Gamma \vdash C$ as follows:

$$\frac{\frac{\mathcal{D}'_1 \quad \frac{}{\Gamma' \vdash a : c} [id]}{\Gamma' \vdash C} [cut] \quad \frac{\mathcal{D}''_1 \quad \frac{}{\Gamma'' \vdash a : c} [id]}{\Gamma'' \vdash C} [cut]}{\Gamma \vdash C} [R]$$

2(b)–2(c). The subcases where $\Gamma, a : A \vdash C$ is an instance of $[ref]$ or $[\perp]$ are similar to the subcase 2(a).

Case 3. $\Gamma \vdash a : A$ and $\Gamma, a : A \vdash C$ are not axiom instances and $a : A$ is not principal in the last rule of \mathcal{D}_1 or \mathcal{D}_2 .

3(a). $\Gamma \vdash a : A$ and $\Gamma, a : A \vdash C$ are not axiom instances and $a : A$ is not principal in the last rule of \mathcal{D}_1 . We consider only the case where the last rule of \mathcal{D}_1 is a two-premises rule:

$$\frac{\mathcal{D}'_1 \quad \mathcal{D}''_1}{\Gamma' \vdash C' \quad \Gamma'' \vdash a : A} [R]$$

If $[R] \neq [\supset_L]$ then $C' \equiv a : A$ holds. By applying the induction hypothesis and the depth-preserving admissibility of weakening and contraction (Proposition 3 and Proposition 6), we obtain a proof of $\Gamma \vdash C$ as follows:

$$\frac{\frac{\mathcal{D}'_1[\Gamma] \quad \mathcal{D}_2[\Gamma']}{\Gamma, \Gamma' \vdash a : A \quad \Gamma, \Gamma', a : A \vdash C} [cut] \quad \frac{\mathcal{D}''_1[\Gamma] \quad \mathcal{D}_2[\Gamma'']}{\Gamma, \Gamma'' \vdash a : A \quad \Gamma, \Gamma'', a : A \vdash C} [cut]}{\Gamma, \Gamma \vdash C} [R]$$

If $[R] = [\supset_L]$ then a proof of $\Gamma \vdash C$ is obtained similarly by applying the induction hypothesis and the depth-preserving admissibility of weakening and contraction as follows:

$$\frac{\mathcal{D}'_1[\Gamma] \quad \frac{\mathcal{D}''_1[\Gamma] \quad \mathcal{D}_2[\Gamma'']}{\Gamma, \Gamma'' \vdash a : A \quad \Gamma, \Gamma'', a : A \vdash C} [cut]}{\Gamma, \Gamma' \vdash C'} [cut]$$

$$\frac{\Gamma, \Gamma' \vdash C' \quad \Gamma, \Gamma'' \vdash C}{\Gamma, \Gamma \vdash C} [\supset_L]$$

3(b). $\Gamma \vdash a : A$ and $\Gamma, a : A \vdash C$ are not axiom instances and $a : A$ is not principal in the last rule of \mathcal{D}_2 . The proof in this case is similar to case 3(a).

Case 4. $a : A$ is principal of the both last rules of \mathcal{D}_1 and \mathcal{D}_2 . Here, we only develop the case of $A \equiv \Box B$. The other cases are similar.

$\Gamma \vdash a : A$ and $\Gamma, a : A \vdash C$ are respectively of the form $\Gamma', a' : \Diamond c' \vdash a : \Box B$ and $\Gamma', a : \Box B, a' : \Diamond c' \vdash C$ with $a \sim a'$ in these sequents and then \mathcal{D}_1 and \mathcal{D}_2 are respectively of the form:

$$\frac{\mathcal{D}'_1}{\Gamma', a : \Diamond c, a' : \Diamond c' \vdash c : B} [\Box_R] \quad \text{and} \quad \frac{\mathcal{D}'_2}{\Gamma', a : \Box B, a' : \Diamond c', c' : B \vdash C} [\Box_L]$$

Using Proposition 7 and Proposition 5 we obtain $\vdash_{\text{G}_{\text{IHL}}}^{\eta_1-1} \Gamma', a' : \Diamond c' \vdash c' : B$. Thus, a proof of $\Gamma \vdash C$ is obtained by using Proposition 3 and the induction hypothesis as follows:

$$\frac{\frac{\frac{\mathcal{D}_1[c' : B]}{\Gamma, a' : \Diamond c', c' : B \vdash a : \Box B} \quad \frac{\mathcal{D}'_2}{\Gamma', a : \Box B, a' : \Diamond c', c' : B \vdash C}}{\Gamma, a' : \Diamond c' \vdash c' : B \quad \Gamma, a' : \Diamond c', c' : B \vdash C} [\text{cut}]}{\Gamma, a' : \Diamond c' \vdash C} [\text{cut}] \quad \square$$

4.3. Another sequent calculus G_{IHL}^2

Now we show that it is possible to define another sequent calculus for IHL without conditions of the form $a \sim b$. We call G_{IHL}^2 the calculus obtained from G_{IHL} by adding the following structural rule:

$$\frac{\Gamma[a/c] \vdash C[a/c]}{\Gamma, c : a \vdash C} [\text{S}]$$

and by replacing $[\text{id}]$, $[\text{ref}]$, $[\Box_L]$ and $[\Diamond_R]$ by the following new rules:

$$\frac{}{\Gamma, a : A \vdash a : A} [\text{id}'] \quad \frac{}{\Gamma \vdash a : a} [\text{ref}']$$

$$\frac{\Gamma, a : \Diamond c, a : \Box A, c : A \vdash C}{\Gamma, a : \Diamond c, a : \Box A \vdash C} [\Box'_L] \quad \frac{\Gamma, a : \Diamond c \vdash c : A}{\Gamma, a : \Diamond c \vdash a : \Diamond A} [\Diamond'_R]$$

Theorem 6. *The sequent calculus G_{IHL}^2 is sound and complete.*

Proof. We show that a sequent is derivable in G_{IHL}^2 if and only if it is derivable in G_{IHL} . \square

5. Decidability of IHL

In this section, we prove the decidability of IHL by using the sequent calculus G_{IHL}^- . The key point of our decision procedure is the use of the cut-elimination property in order to provide a suitable subformula property different from the usual one, called the *quasi-subformula property*.

We introduce a notion of *redundancy* satisfying the fact that any sequent valid in IHL has an *irredundant proof*. Then we prove that there is no infinite derivation which is not irredundant and deduce the decidability result.

5.1. Introduction of new nominals

In order to prove the decidability of IHL by using G_{IHL}^- , we must solve the problem of the introduction of new nominals using $[\Diamond_L]$ and $[\Box_R]$. This problem is similar to the one of the introduction of new labels in the labelled sequent calculi of the intuitionistic modal logics studied by Simpson in [17]. Let us note that the introduction of new nominals or labels is not a problem in the case of classical modal and hybrid logics because we can define proof systems with only invertible rules allowing terminating proof-search [5,15].

In the case of Simpson's calculi, the problem of the introduction of new labels was resolved using the following property:

for any derivation, there is a positive integer n such that there is no sequent containing a chain of length greater than n . A chain is a sequence of the form $x_0 R x_1, \dots, x_{m-1} R x_m$ where R represents the accessibility relation, x_0 is not a new label and x_{i+1} is a new label for $i \in [0, m-1]$.

We can say that any infinite derivation is redundant because there are necessarily two sequents where one can be obtained from the other by renaming some new nominals (for more details see [17]). Similarly, for G_{IHL} a chain is a sequence of the form $a_0 : \Diamond a_1, a_1 : \Diamond a_2, \dots, a_{m-1} : \Diamond a_m$ where a_0 is not a new nominal and a_{i+1} is a new nominal for $i \in [0, m-1]$.

The following example shows that the previous property fails in the case of G_{IHL}^- :

$$\begin{array}{c}
 \vdots \\
 a_0 : \Diamond a_1, \dots, a_i : \Diamond a_{i+1}, a_1 : a_0, \dots, a_i : a_0, a_2 : p, \dots, a_{i+1} : p, a_0 : \Box(a_0 \wedge \Diamond p) \vdash a_0 : \perp \\
 \vdots \\
 \vdots \\
 \hline
 a_0 \Diamond a_1, a_1 : \Diamond a_2, a_2 : \Diamond a_3, a_1 : a_0, a_2 : a_0, a_2 : p, a_3 : p, a_0 : \Box(a_0 \wedge \Diamond p) \vdash a_0 : \perp \\
 \hline
 a_0 : \Diamond a_1, a_1 : \Diamond a_2, a_1 : a_0, a_2 : a_0, a_2 : p, a_2 : \Diamond p, a_0 : \Box(a_0 \wedge \Diamond p) \vdash a_0 : \perp \quad [\Diamond_L] \\
 \hline
 a_0 : \Diamond a_1, a_1 : \Diamond a_2, a_1 : a_0, a_2 : p, a_2 : a_0 \wedge \Diamond p, a_0 \Box(a_0 \wedge \Diamond p) \vdash a_0 : \perp \quad [\wedge_L] \\
 \hline
 a_0 \Diamond a_1, a_1 \Diamond a_2, a_1 : a_0, a_2 : p, a_0 \Box(a_0 \wedge \Diamond p) \vdash a_0 : \perp \quad [\Box_L] \\
 \hline
 a_0 \Diamond a_1, a_1 \Diamond a_2, a_1 : a_0, a_2 : p, a_0 \Box(a_0 \wedge \Diamond p) \vdash a_0 : \perp \quad [\Diamond_L] \\
 \hline
 a_0 : \Diamond a_1, a_1 : a_0, a_1 : \Diamond p, a_0 \Box(a_0 \wedge \Diamond p) \vdash a_0 : \perp \quad [\wedge_L] \\
 \hline
 a_0 : \Diamond a_1, a_1 : a_0 \wedge \Diamond p, a_0 : \Box(a_0 \wedge \Diamond p) \vdash a_0 : \perp \quad [\Box_L] \\
 \hline
 a_0 : \Diamond a_1, a_0 : \Box(a_0 \wedge \Diamond p) \vdash a_0 : \perp
 \end{array}$$

Moreover, it is easy to see that, in this infinite derivation, there are no two distinct sequents such that one can be obtained from the other by renaming some new nominals. To solve this problem, we associate to every sequent S appearing in a given derivation a particular sequent, called the *equivalid* sequent of S , satisfying the previous property and the fact that a sequent has a proof if and only if its *equivalid* sequent has a proof of the same size. The *equivalid* sequents are obtained by renaming some new nominals.

5.2. Quasi-subformula property

Let us recall that the *subformulas* of a formula A are inductively defined as follows:

- A is a subformula of A ;
- if $B \otimes C$ is a subformula of A then so are B, C , for $\otimes = \wedge, \vee, \supset$;
- if $\Box B$ is a subformula of A then so is B , for $\Box = \Box, \Diamond$;
- if $a : B$ is a subformula of A then so is B .

Now, we introduce the notion of *quasi-subformula*. It is similar to the weak subformula notion introduced in [15] and the quasi-subformula notion given in [7].

Definition 5 (*Quasi-subformula*). Let A be a formula, the *quasi-subformulas* of A are inductively defined as follows:

- for every $\otimes \in \{\wedge, \vee, \supset\}$, the quasi-subformulas of $a : A \otimes B$ are $a : A \otimes B$ and all the quasi-subformulas of $a : A$ and $a : B$;
- the quasi-subformulas of $a : \Box A$ for $\Box \in \{\Diamond, \Box\}$ are $a : \Box A$ and all the quasi-subformulas of $c : A$ for an arbitrary c .

Theorem 7 (*Quasi-subformula property*). Let S be a sequent and \mathcal{D} be a derivation of S in G_{IHL}^- . Any formula occurrence $a : A$ in \mathcal{D} is either a quasi-subformula of a formula in S or of the form $a : \Diamond c$.

Proof. By induction on the depth of \mathcal{D} . We only have to distinguish the cases of the last rule application. \square

We are interested in the size of derivations. Previously, we proved that weakening and contraction are depth-preserving admissible for G_{IHL}^- . Weakening and contraction are also size-preserving admissible for G_{IHL}^- . Moreover, if a sequent S has a proof in G_{IHL}^- of size n , then any sequent obtained from S by renaming some nominals with others has a proof of size smaller or equal to n .

Proposition 10. If $\Gamma \vdash C$ has a proof in G_{IHL}^- of size n , then $\Gamma, a : A \vdash C$ has a proof of size smaller or equal to n .

Proof. Similar to the proof of Proposition 3. \square

Proposition 11. If $\Gamma, a : A, a : A \vdash C$ has a proof in G_{IHL}^- of size n , then $\Gamma, a : A \vdash C$ has a proof of size smaller or equal to n .

Proof. Similar to the proof of Proposition 6. \square

Proposition 12. Let $\Gamma \vdash C$ be a sequent and $f : \text{Nom}(S) \rightarrow M$ be a renaming function. If $\Gamma \vdash C$ has a proof in G_{IHL}^- of size n , then $f(\Gamma \vdash C)$ has a proof of size smaller or equal to n .

Proof. Similar to the proof of Proposition 5. \square

Let S' be a sequent appearing in a given derivation of the sequent S , \sim be the associated equivalence relation of S' and let c be an element of $\text{Nom}(S')$ such that $c \notin \text{Nom}(S)$. We denote $N(c, S')$ the set of nominals defined by $a \in N(c, S')$ if and only if $a \in \text{Nom}(S)$ and $a \sim c$.

Definition 6. Let S' be a sequent appearing in a given derivation of a sequent S and $N = \text{Nom}(S)$ with an order fixed on the elements of N . We define $\text{Eq}(S')$ as the sequent $f(S')$ where f is a renaming function defined as follows

$$f(c) = \begin{cases} c & \text{if either } c \in N \text{ or } N(c, S') = \emptyset, \\ c' & \text{otherwise, with } c' = \max(N(c, S')) \end{cases}$$

where \max denotes the maximum.

We can see that any formula of $\text{Eq}(S')$ is a quasi-subformula of a formula of S or of the form $a : \Diamond c$. This comes from the quasi-subformula property.

Proposition 13. Let S' be a sequent appearing in a given derivation. S' has a proof in G_{IHL}^- of size n if and only if $\text{Eq}(S')$ has a proof of size n .

Proof. We only have to prove that for any sequent S and for all $a, b \in \text{Nom}(S)$ such that $a \sim b$, S has a proof in G_{IHL}^- of size n if and only if $S[a/b]$ has a proof in G_{IHL}^- of size n . The *if part* is proved by induction on n . The *only if part* comes from Proposition 12. \square

5.3. N -chains

Now, we introduce the notion of N -chain. Intuitively, the N -chains correspond to sequences of formulas of the form $a : \Diamond c$ which will allow us to give a description of the arrangement of the new nominals introduced using the rules $[\Box_R]$ and $[\Diamond_L]$. The key point is that the length of these sequences, in any sequent equivalent to a sequent appearing in any derivation of S in G_{IHL}^- , is bounded by the nesting degree of S .

The nesting degree of a formula A , denoted $\text{nest}(A)$, is inductively defined as follows:

- $\text{nest}(p) = 0$; $\text{nest}(a) = 0$; $\text{nest}(\perp) = 0$;
- $\text{nest}(A \otimes B) = \max(\text{nest}(A), \text{nest}(B))$ where $\otimes \in \{\wedge, \vee, \supset\}$;
- $\text{nest}(a : A) = \text{nest}(A)$;
- $\text{nest}(\Box A) = 1 + \text{nest}(A)$ where $\Box \in \{\Box, \Diamond\}$.

The nesting degree of a sequent is the maximum of the nesting degrees of its formulas.

Definition 7 (N -chain). Let $S = \Gamma \vdash C$ be a sequent and N a finite set of nominals. An N -chain is a sequence of the form $a_0 : \Diamond a_1, a_1 : \Diamond a_2, \dots, a_{k-1} : \Diamond a_k$ (a_0 when $k = 0$) where

- $a_{i-1} : \Diamond a_i \in \Gamma$ for $i = 1, \dots, k$;
- $a_0 \in N$;
- $a_i \notin N$ for $i = 1, \dots, k-1$;
- if $a_k \in N$ and $k \neq 0$ then $k > 1$; and
- if $a_k \notin N$ then there is no nominal a such that $a_k : \Diamond a \in \Gamma$.

Proposition 14. Let S and S' be two sequents such that S' is the equivalent sequent of a sequent appears in a derivation of S . Then, for all $a : \Diamond b \in \Gamma$ ($S' = \Gamma \vdash C$) such that either $a \notin \text{Nom}(S)$ or $b \notin \text{Nom}(S)$, $a : \Diamond b$ belongs to a $\text{Nom}(S)$ -chain of S' .

Moreover, the length of any $\text{Nom}(S)$ -chain in S' is smaller or equal to $\text{nest}(S) + 1$.

Proof. The first property is obtained from the rules that introduce new nominals, namely $[\Box_L]$ and $[\Diamond_R]$, and also from the definition of the equivalent sequents where only some nominals introduced using the previous two rules are renamed by nominals in $\text{Nom}(S)$.

The second property holds because if there exists a $\text{Nom}(S)$ -chain with a length greater than $\text{nest}(S)$ in S' then there is a nominal $b \notin \text{Nom}(S)$ in this $\text{Nom}(S)$ -chain and another nominal $a \in \text{Nom}(S)$ such that $a \sim b$ and b is renamed by a in order to build $\text{Eq}(S')$ (from the definition of the rules $[\Box_L]$ and $[\Diamond_R]$). \square

A preorder, denoted \lesssim_S , on the sequents appearing in the derivations of S is defined as follows: $S_1 \lesssim_S S_2$ if and only if there exists a renaming function f such that $\text{set}(f(\Gamma_1)) \subseteq \text{set}(\Gamma_2)$ and $f(C_1) = C_2$ where $\text{Eq}(S_1) = \Gamma_1 \vdash C_1$ and $\text{Eq}(S_2) = \Gamma_2 \vdash C_2$.

We use $\text{set}(\Gamma)$ to denote the set underlying the multiset Γ (the set of the formulas of Γ). Moreover, we use the notation $\text{set}(\Gamma \vdash C)$ for $\text{set}(\Gamma) \cup C$.

Proposition 15. Let S_1 and S_2 be two sequents in a derivation of S . If $S_1 \lesssim_S S_2$ then if S_1 has a proof of size n then S_2 has a proof of size smaller or equal to n .

Proof. From the size-preserving admissibility of weakening and contraction (see Proposition 10 and Proposition 11) and also Proposition 12. \square

5.4. Trees and skeletons

Now, we use the notion of N -chain to represent the equivalent sequents by sets of *trees*. Then we derive from such trees other trees called *skeletons*. Next, we prove that the numbers of nodes of the skeletons obtained from the equivalent sequents of the sequents in a given derivation are bounded. Using this property, we show that for any derivation of S having an infinite branch, there are two sequents S' and S'' , with S' strictly occurring above S'' in this branch, and such that $S' \lesssim_S S''$.

Definition 8 (Tree). Let S and $S' = \Gamma \vdash c : C$ be two sequents such that S' is the equivalent sequent of a sequent in a derivation of S and a be an element of $\text{Nom}(S)$. We define the *tree associated to a in S'* , denoted $T(a, S')$, as follows:

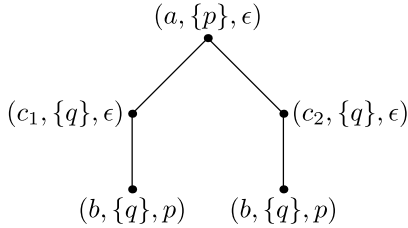
- The nodes are labelled with triples $(b, \Gamma(b), \alpha(b))$ where $b \in \text{Nom}(S')$, $\Gamma(b) = \{A \mid b : A \in \Gamma\}$ and

$$\alpha(b) = \begin{cases} C & \text{if } b = c, \\ \epsilon & \text{otherwise} \end{cases}$$

where ϵ is a symbol not in Nom and Prop .

- The root node is labelled with $(a, \Gamma(a), \alpha(a))$.
- A node labelled with (b', Δ', α') is an immediate successor of a node labelled with (b, Δ, α) iff $b : \Diamond b'$ belongs to a $\text{Nom}(S)$ -chain of S' starting with a .

Let S' be the sequent $a : p, c_1 : q, c_2 : q, b : q, a : \Diamond c_1, a : \Diamond c_2, c_1 : \Diamond b, c_2 : \Diamond b \vdash b : p$ which is an equivalent sequent of a sequent appearing in a derivation of S such that $\text{Nom}(S) = \{a, b\}$. $T(a, S')$ is the following tree:



We characterize every sequent $S' = \Gamma \vdash C$ which is equivalent to a sequent appearing in a derivation of S by the set of trees $T_{S'} = \{T(a, S') \mid a \in \text{Nom}(S)\}$. It is called the *tree set characterizing S'* . From this set we can easily obtain the value of $\text{set}(\Gamma \vdash C)$.

We write \cong_N , with N is a finite set of nominals, the equivalence relation on the trees defined by $T_1 \cong_N T_2$ if and only if $T_1 = f_1(T_2)$ and $T_2 = f_2(T_1)$ where f_1 and f_2 are two renaming functions satisfying the property that for all $a \in N$ we have $f_1(a) = f_2(a) = a$, i.e., two trees are equivalent if and only if each tree can be obtained from the other by renaming some nominals which is not in N .

Let us note that any tree can be represented by the expression (r, L) where r is the root node and L is a list of trees. The set of the *subtrees* of a tree T is inductively defined as follows: T is a subtree of T ; if (r, L) is a subtree of T then so are the elements of L . We note $\text{dep}(T)$ the depth of the tree T .

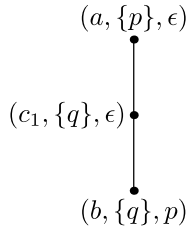
Definition 9 (Skeleton). Let S and S' be two sequents such that S' is the equivalent sequent of a sequent in a derivation of S , T be an element of the tree set characterizing S' and $N = \text{Nom}(S)$.

A *skeleton* of T , denoted $\text{Sk}(T)$, is a tree built from T in a following way:

- *Step 0:* we initialize $\text{Sk}(T)$ with T .
- *Step $i + 1$:* for all subtrees $ST = (r, L)$ of $\text{Sk}(T)$ of depth equal to $(i + 1)$, we replace ST in $\text{Sk}(T)$ by (r, L') where L' is a sublist of L obtained as follows:

- we start with $L' = []$;
- for all $\mathcal{T}_0 \in L$, if there is no tree \mathcal{T}_1 in L' such that $\mathcal{T}_0 \cong_N \mathcal{T}_1$, then we add \mathcal{T}' to L' .

A skeleton of the tree given in the previous example is:



We can see that there is not always a single skeleton associated to a tree. However, all the skeletons associated to any tree are equivalent. If $\{T_1, \dots, T_k\}$ is the tree set characterizing $\mathcal{S} = \Gamma \vdash C$ and $\mathcal{S}' = \Gamma' \vdash C$ is the sequent obtained from $\{Sk(T_1), \dots, Sk(T_k)\}$ then $\Gamma' \subseteq \Gamma$ and $set(\mathcal{S}')$ can be obtained from $set(\mathcal{S})$ by renaming some new nominals.

Proposition 16. *Let \mathcal{S} be a sequent and \mathcal{D} be a derivation of \mathcal{S} . Then there exists a constant K such that for any sequent \mathcal{S}' equivalent to a sequent appearing in \mathcal{D} , if \mathcal{T} is in the tree set characterizing \mathcal{S}' then the number of nodes of $Sk(\mathcal{T})$ is smaller or equal to K .*

Proof. We know that the depth of \mathcal{T} is equal to the depth of $\mathcal{T}' = Sk(\mathcal{T})$ and is smaller or equal to $nest(\mathcal{S}) + 1$ (Proposition 14). Let Φ be the set of the subformulas of the formulas of \mathcal{S} and ϕ its size. The size of the set of the subsets of Φ is 2^ϕ . Using the quasi-subformula property (Theorem 7), we prove that for all n a node of \mathcal{T}' of depth $dep(\mathcal{T}') - 1$, n has at most $K_1 = (N + 1) \times 2^\phi \times 2$ successors where N is the size of $Nom(\mathcal{S})$.

Similarly, the number of the successors of any node in \mathcal{T} of depth $dep(\mathcal{T}') - 2$ is at most equal to $K_2 = (N + 1) \times 2^{K_1} \times 2$. We continue until the root node ($K_{dep(\mathcal{T})}$). Thus, we can take the constant K equal to

$$1 + \sum_{i=0}^{dep(\mathcal{T})-1} \prod_{j=0}^i K_{dep(\mathcal{T})-j} \quad \square$$

Proposition 17. *Let \mathcal{S} be a sequent and \mathcal{D} be a derivation of \mathcal{S} . The set of skeletons obtained from the equivalent sequents of the sequents in \mathcal{D} is partitioned into a finite set of equivalence classes by $\cong_{Nom(\mathcal{S})}$.*

Proof. A consequence of Proposition 16. \square

Proposition 18. *Let \mathcal{D} be a derivation of a sequent \mathcal{S} with an infinite branch $\mathcal{B} = (\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_k, \dots)$. Then, there exist i and j such that $i < j$ and $\mathcal{S}_j \lesssim_{\mathcal{S}} \mathcal{S}_i$.*

Proof. Let $\mathcal{B}' = (\mathcal{S}'_1, \mathcal{S}'_2, \dots, \mathcal{S}'_i, \dots)$ where $\mathcal{S}'_i = Eq(\mathcal{S}_i)$ for $i = 1, 2, \dots$. We associate to every sequent in \mathcal{B}' the set of tree $T_{\mathcal{S}'_i} = \{Sk(T(a, \mathcal{S}'_i)) \mid a \in Nom(\mathcal{S})\}$. Using Proposition 17, we deduce that there exist two sequents $\mathcal{S}'_i = \Gamma'_i \vdash C'_i$ and $\mathcal{S}'_j = \Gamma'_j \vdash C'_j$ such that $i < j$ and for all $T_j \in T_{\mathcal{S}'_j}$, there is $T_i \in T_{\mathcal{S}'_i}$ satisfying $T_i \cong_{Nom(\mathcal{S})} T_j$. If $\Gamma''_i \vdash C''_i$ and $\Gamma''_j \vdash C''_j$ are the two sequents obtained respectively from $T_{\mathcal{S}'_i}$ and $T_{\mathcal{S}'_j}$, then there is a renaming function f such that $f(\Gamma''_j) = \Gamma''_i$ and $f(C''_j) = C''_i$. Moreover, we have $\Gamma''_i \subseteq \Gamma'_i$ and $C''_i = C'_i$. Since there is a renaming function g such that $set(g(\Gamma''_j)) = \Gamma''_j$ and $g(C''_j) = C''_j$, $set((f \circ g)(\Gamma''_j)) \subseteq set(\Gamma'_i)$ and $(f \circ g)(C''_j) = C'_i$ hold. Therefore, we deduce that $\mathcal{S}_j \lesssim_{\mathcal{S}} \mathcal{S}_i$. \square

5.5. A decision procedure for IHL

Now we introduce a notion of *redundancy* on cut-free derivations in our calculus such that any sequent that is valid has an irredundant proof. Then, by using the quasi-subformula property, we prove that there is no infinite proof which is redundant and then provide a decision procedure for IHL and then prove the decidability of this logic through proof-search using our sequent calculus.

Definition 10. A derivation of \mathcal{S} is *redundant* if it contains two sequents \mathcal{S}' and \mathcal{S}'' , with \mathcal{S}' occurring strictly above \mathcal{S}'' in the same branch, such that $\mathcal{S}' \lesssim_{\mathcal{S}} \mathcal{S}''$. A derivation is *irredundant* if it is not redundant.

Proposition 19. *If \mathcal{S} is valid in IHL then it has an irredundant proof in $\mathbf{G}_{\text{IHL}}^-$.*

Proof. By induction on the size s of the proof of \mathcal{S} .

If $s = 1$ then it is an irredundant proof.

Now, we assume that for any sequent, if it has a proof of size smaller or equal to n ($n \geq 1$), then it has an irredundant proof (the induction hypothesis).

Let \mathcal{D} be a proof of \mathcal{S} of size $(n + 1)$. If \mathcal{D} is irredundant then we have the result. Otherwise it has a branch containing two sequents \mathcal{S}_1 and \mathcal{S}_2 such that \mathcal{S}_1 occurring above \mathcal{S}_2 and $\mathcal{S}_1 \lesssim_{\mathcal{S}} \mathcal{S}_2$. Let n' be the size of the subderivation of \mathcal{S}_1 in \mathcal{D} . It is easy to see that the size of the subderivation of \mathcal{S}_2 in \mathcal{D} is strictly greater than n' . Using Proposition 15, we know that \mathcal{S}_2 has a proof \mathcal{D}_2 of size smaller or equal to n' . So by replacing in \mathcal{D} the subderivation of \mathcal{S}_2 with \mathcal{D}_2 we obtain a proof of \mathcal{S} of size smaller or equal to n . Therefore, by applying the induction hypothesis, we deduce that \mathcal{S} has an irredundant proof. \square

Now, we provide a decision procedure for the sequents in IHL based on the redundancy notion similar to this proposed in [17] for the intuitionistic modal logics. It consists of an exhaustive search for an irredundant derivation.

Let \mathcal{S} be a sequent.

- *Step 1:* we start with the derivation containing only \mathcal{S} which is the unique irredundant derivation of size 1. If this derivation is a proof then we return it. Otherwise we move to the next step.
- *Step $i + 1$:* we build the set of all the irredundant derivations of size $i + 1$. If this set contains a proof of \mathcal{S} then we return it. Otherwise if this set is empty then the decision algorithm fails, else we move to the next step.

There are only a finite number of possible rule applications (the choice of the new nominals introduced by the rules $[\Box_R]$ and $[\Diamond_L]$ is not essential). Thus, the set of the irredundant derivations of size $i + 1$ is finite. Moreover, this set can be built in a finite time because the $\lesssim_{\mathcal{S}}$ relation is decidable.

Theorem 8 (Decidability). *The logic IHL is decidable.*

Proof. Using Proposition 18, we know that there is no infinite irredundant derivation. Thus, we deduce that our algorithm terminates. Therefore, IHL is decidable. \square

6. Conclusion

In this work, we provide the first sequent calculus for the hybrid intuitionistic logic IHL [7] that is appropriate for proof-search thanks to the absence of structural rules. After proving the main properties of this calculus that are soundness, completeness and cut-elimination, we define a decision procedure and then we propose the first proof of decidability of this logic. The study of complexity of IHL [1] will be the next step developed in further works but we will also consider extensions of our calculi with rules corresponding to conditions on the accessibility relations (geometric theories) like reflexivity, symmetry and transitivity, in order to obtain a system in which each condition on the accessibility relation has a corresponding rule and each combination of these rules is complete for the logic with the corresponding conditions.

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